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# Finite-size logarithmic corrections in the free energy of the mean spherical model

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**Abstract.** The validity of the finite-size scaling prediction about the existence of logarithmic corrections in the free energy due to corners is studied by the example of the mean spherical model. The general case of a hypercubic lattice of arbitrary dimensionality $d>2$, under boundary conditions which are periodic in $d'\geqslant 0$ dimensions and free in the remaining $d-d'$ dimensions is considered. The critical regime, as the size of the system $L\to\infty$, is specified by the asymptotic behaviour of the ratio $L/\xi_L$, where $\xi_L$ is the correlation length of the finite system. New results are the double-logarithmic corrections due to corners and logarithmic corrections due to one-dimensional edges in the regime $L/\xi_L\propto\ln L$ which takes place at the bulk critical point.

## 1. Introduction

As was shown by Cardy and Peschel [1], the free energy of two-dimensional conformally invariant models on manifolds with Euler number $\chi=0$ and non-smooth boundaries contains logarithmic finite-size corrections, arising from each corner, with universal amplitudes proportional to the conformal anomaly number. By using finite-size scaling arguments Privman [2] has predicted similar corrections for systems of linear size $L$ at arbitrary dimensionality $d$

$$\Delta F^{(c)}=\sum_{\text{corners}} y_i \ln L \tag{1.1}$$

with some universal amplitudes $y_i$ attributed to each corner.

The extension to higher dimensionality of the prediction (1.1) has been checked by the example of a few models. Logarithmic contributions in the free energy, $\Delta F^{(1)}$, at arbitrary dimensionality and three different types of boundary conditions, have been obtained in a Gaussian-type model by Gelfand and Fisher [3, 4]. These contributions were found to arise in the small block limit due to the zero-eigenvalue mode and, contrary to the predictions of both conformal theory [1] and finite-size scaling arguments [2], persist under periodic boundary conditions. It should be mentioned that $\ln L$ terms, both under *free* and *periodic* boundary conditions, have been found by Duplantier and David [5] in the free energy of the two-dimensional conformally invariant spanning tree model.

Logarithmic finite-size corrections have been found also in the exactly solvable for general $d$ constrained monomer-dimer model (CMD) [6]. It has been shown [7] that the difference in the free energies $F_{d,0}-F_{d,d'}$ of the $d$-dimensional CMD model, with $F_{d,0}$ corresponding to fully free boundary conditions, and $F_{d,d'}$ corresponding to periodic boundaries in $d'\geq 1$ dimensions and free boundaries in the remaining $d-d'$ dimensions, contains the logarithmic correction term $2^{1-d}\ln L$ which comes from the $2^d$ corners of the system with linear size $L$.

The aim of the present work is to study the problem by the example of the mean spherical model [8]. In section 2 we formulate the model with an emphasis on its similiarities and dissimilarities with the Gaussian and CMD models. Short comments on this aspect are given at the end of each of the following sections. Section 3 describes shortly the mathematical transformations used to obtain a convenient representation for the partition function. The detailed finite-size analysis of the free energy is given in section 4 for arbitrary dimensionalities. The discussion part, section 5, is devoted to the classification of the different logarithmic in $L$ contributions. By using the results of [10] for the finite-size asymptotic behaviour of the correlation length in the three-dimensional case, we obtain also explicit expressions for the corner and edge contributions.

## 2. The problem

We consider the ferromagnetic mean spherical model on a finite $d$ dimensional hypercubic lattice $\Lambda_d=L_1\times\cdots\times L_d\in\mathbb{Z}^d$ of $N$ sites, with periodic boundary conditions in the first $d'\geq 0$ dimensions and free boundaries in the remaining $d'-d$ dimensions. Assuming nearest-neighbours interaction, the partition functions of both the mean spherical and Gaussian models are defined in terms of the quadratic form

$$s\sum_{i\in\Lambda}\sigma_i^2-\tfrac{1}{2}K\sum_{\langle i,j\rangle}\sigma_i\sigma_j \tag{2.1}$$

where $\sigma_i\in\mathbb{R}^1$, $i\in\Lambda$, are the dynamical variables, $s>0$ is the spherical field, and $K>0$ is the dimensionless coupling. The summation in the interaction term is taken over all different pairs $\langle i,j\rangle$ of nearest neighbours under the imposed boundary conditions. The eigenvalues $\varepsilon_{d,d'}$ of the quadratic form (2.1) are well known [3]; they can be labelled by the set $\boldsymbol{k}=\{k_1,\ldots,k_d\}$ with

$$k_\nu=0,1,\ldots,L_\nu-1 \qquad \nu=1,\ldots,d. \tag{2.2}$$

It is convenient to replace the spherical field $s$ by the correlation length $\xi$ of the Gaussian model, defined as $\xi^{-2}=2s/K-2d>0$, and rewrite the eigenvalues $\varepsilon_{d,d'}$ of (2.1) in the form

$$\varepsilon_{d,d'}(\boldsymbol{k};\xi,K;L)=(K/2)[\xi^{-2}+\lambda_{d,d'}(\boldsymbol{k};L)] \tag{2.3}$$

where

$$\lambda_{d,d'}(\boldsymbol{k};L)=2\sum_{\nu=1}^{d'}\left[1-\cos\frac{2\pi k_\nu}{L_\nu}\right]+2\sum_{\nu=d'+1}^{d}\left[1-\cos\frac{\pi k_\nu}{L_\nu}\right]. \tag{2.4}$$

For simplicity of notation, we assume in the remainder that $L_1=\cdots=L_d=L$.

The evaluation of the partition function $Z^G_{d,d'}(K,\xi;L)$ of the Gaussian model now yields

$$Z^G_{d,d'}(K,\xi;L)=(2\pi/K)^{N/2}(\tilde{Q}_{d,d'}(\xi;L))^{-1/2} \tag{2.5}$$

where

$$\tilde{Q}_{d,d'}(\xi;L)=\prod_{\nu=1}^{d}\prod_{k_\nu=0}^{L_\nu-1}[\xi^{-2}+\lambda_{d,d'}(\boldsymbol{k};L)]. \tag{2.6}$$

Let us introduce the sets $D=\{1,\ldots,d\}$ and $D'=\{1,\ldots,d'\}\subseteq D$ if $d'\geqslant 1$, $D'=\varnothing$ if $d'=0$. Let $|S|$ be the number of elements in a set $S\subseteq D$. Note that for any set $S\subseteq D$.

$$\lambda_{d,d'}(\boldsymbol{k};L)\Big|_{k_\mu=0,\,\mu\in D\setminus S}=\lambda_{n,m}(\boldsymbol{k};L) \tag{2.7}$$

where

$$n=|S| \qquad m=|S\cap D'|. \tag{2.8}$$

Therefore, the product (2.6) can be split into co-factors, each of which corresponds to a given set $D/S$ of zero-valued indices $k_\mu$, $\mu\in D\setminus S$,

$$\tilde{Q}_{d,d'}(\xi;L)=\xi^{-2}\prod_{\substack{S\subseteq D\\ S\neq\varnothing}}Q_{n,m}(\xi;L) \tag{2.9}$$

where

$$Q_{n,m}(\xi;L)=\prod_{\nu\in S}\prod_{k_\nu=1}^{L-1}[\xi^{-2}+\lambda_{n,m}(\boldsymbol{k};L)]. \tag{2.10}$$

Obviously, the factor $\xi^{-2}$ in (2.9) represents the contribution of the zero-eigenvalue mode with $k_\nu=0$, $\nu=1,\ldots,d$. After substitution in (2.5) it leads to a factor of $\xi$ in the finite-size partition function of the Gaussian model, thus causing the divergence of the latter at criticality, i.e. at $\xi=\infty$. Dropping the zero-mode contribution in (2.9) and passing to the limit $\xi\to\infty$ one obtains the partition function $\Lambda_{d,d'}$ of the critical CMD model [7]:

$$\Lambda_{d,d'}(L)=\prod_{\substack{S\subseteq D\\ S\neq\varnothing}}Q_{n,m}(\infty;L). \tag{2.11}$$

Therefore, as noted in [7], the logarithmic in $L$ contributions in the free energy $-\ln\Lambda_{d,d'}(L)$ of the CMD model are proportional, with the coefficient $-2$, to those of the critical Gaussian model in which the $\boldsymbol{k}=0$ mode has been removed 'by hand'.

The canonical free energy of the mean spherical model, $F_{d,d'}$, normalized by the factor $(kT)^{-1}$ and up to unessential additive terms, is given by the Legendre transformation

$$F_{d,d'}(K;L)=\sup_{\xi}\tfrac{1}{2}\{\ln\tilde{Q}_{d,d'}(\xi;L)-\xi^{-2}KN\}. \tag{2.12}$$

The point $\xi=\xi_L(K)$ at which the supremum in (2.12) is attained defines the finite-size correlation length $\xi_L(K)$ of the mean spherical model. Since the

$L$-dependence of $\xi_L(K)$ may be non-trivial and sensitive to the critical regime under consideration, the finite-size corrections in the fee energy of the mean spherical model may essentially differ from those for the Gaussian and CMD models.

## 3. Mathematical techniques

The products $Q_{n,m}(\xi; L)$, see (2.9) and (2.10), which enter into expression (2.12) for the free energy, will be evaluated with the aid of techniques similar to the one developed in [7] for the constrained monomer-dimer model. The mathematical transformations involved are based on the identities

$$\prod_{k=1}^{L-1} [a^2+2-2\cos(2\pi k/L)] = a^{-2} g_L^{(0)}(a) \tag{3.1}$$

$$\prod_{k=1}^{L-1} [a^2+2-2\cos(\pi k/L)] = a^{-1} g_L^{(1)}(a) \tag{3.2}$$

where

$$g_L^{(0)}(a) = \{2^{-L}[(a^2+4)^{1/2}+a]^L - 2^L[(a^2+4)^{1/2}+a]^{-L}\}^2 \tag{3.3}$$

$$g_L^{(1)}(a) = (a^2+4)^{-1/2}\{2^{-2L}[(a^2+4)^{1/2}+a]^{2L} - 2^{2L}[(a^2+4)^{1/2}+a]^{-2L}\}. \tag{3.4}$$

These are simple consequences of a known trigonometric identity [9], valid for any integer $L>1$ and real $a \neq 0$.

The idea is, by using substitutions of the form $a^2 = \xi^{-2} + \lambda_{n-1,m-1}(k; L)$ in (3.1) or (3.2), to generate chains of recurrent relations for the products $Q_{n,m}(\xi; L)$. These chains terminate either at (if $n=m$)

$$Q_{1,1}(\xi; L) = \prod_{k=1}^{L-1} \left[\xi^{-2}+2-2\cos\frac{2\pi k}{L}\right] = \xi^2 g_L^{(1)}(\xi^{-1}) \tag{3.5}$$

or at (if $n>m$)

$$Q_{1,0}(\xi; L) = \prod_{k=1}^{L-1} \left[\xi^{-2}+2-2\cos\frac{\pi k}{L}\right] = \xi g_L^{(1)}(\xi^{-1}). \tag{3.6}$$

For example, in the case $n>m$ we obtain

$$Q_{n,m}(\xi; L) = (Q_{1,0}(\xi; L))^{(-1)^m(-2)^{1-n+m}} \prod_{j=1}^{n-m-1} (R^{(1)}_{n-m-j,0}(\xi; L))^{(-1)^m(-2)^{1-j}}$$

$$\times \prod_{q=1}^{m} (R^{(0)}_{n-q,m-q}(\xi; L))^{(-1)^{q-1}} \tag{3.7}$$

where, $n=|S| \geqslant 1$, $m=|S \cap D'| \geqslant 0$

$$R^{(\tau)}_{n,m}(\xi; L) = \prod_{\nu \in s} \prod_{k_\nu=1}^{L-1} g_L^{(\tau)}([\xi^{-2}+\lambda_{n,m}(k; L)]^{1/2}) \qquad \tau=0,1. \tag{3.8}$$

The above-described transformations lead to the following final expression, valid for arbitrary intger values of the dimensionalities $d$ and $d'$:

$$\ln \tilde{Q}_{d,d'}(\xi; L) = -2\ln\xi + \sum_{\substack{S\subseteq D\\ n>m}} 2^m(-\tfrac{1}{2})^{n-1}\ln Q_{1,0}(\xi; L)$$

$$+\sum_{\substack{S\subseteq D'\\ n>0}} (-1)^{n-1}\ln Q_{1,1}(\xi; L) + \sum_{\substack{S\subseteq D'\\ n>1}}\sum_{q=1}^{n-1}(-1)^{q-1}\ln R^{(0)}_{n-q,n-q}(\xi; L)$$

$$+\sum_{\substack{S\subseteq D\\ n>m+1}}\sum_{q=1}^{n-m-1}(-1)^m(-\tfrac{1}{2})^{q-1}\ln R^{(1)}_{n-m-q,0}(\xi; L)$$

$$+\sum_{\substack{S\subseteq D\\ n>m>0}}\sum_{q=1}^{m}(-1)^{q-1}\ln R^{(0)}_{n-q,m-q}(\xi; L). \tag{3.9}$$

Here $n=|S|\geqslant 1$, $m=|S\cap D'|$ and the convention is that a sum over an empty set is zero: in the case of fully free boundary conditions, $d'=0$, $D'=\varnothing$ and $m=0$, therefore the third, fourth and sixth term in the right-hand side of (3.9) vanish; in the case of fully periodic boundary conditions, $d'=d$, $D'=D$ and $m=n$, hence the second, fifth and sixth term in the right-hand side of (3.9) vanish.

Note that taking the limit $\xi\to\infty$ in (3.5) and (3.6) one obtains $Q_{1,1}(\infty; L)=L^2$ and $Q_{1,0}(\infty; L)=L$, respectively, i.e. the results used in the analysis of the CMD model (compare equation (3.7) with equation (3.19) in [7]). In the same limit $R^{(0)}_{n,m}(\infty; L)$ and $R^{(1)}_{n,0}(\infty; L)$ are easily shown not to contribute logarithmically in $L$ terms. In the mean spherical model, however, the situation is much more complicated due to the specific $L$-dependence of the finite-size correlation length $\xi_L(K)$ in the different critical regimes. A finite-size an analysis of expression (3.10) when $\xi\to\infty$ as $L\to\infty$ will be given in the next section.

## 4. Logarithmic finite-size corrections in the free energy

As already mentioned, the mean spherical model differs from the Gaussian model by the fact that the correlation length $\xi_L(K)$ is not a free parameter. At the first stage of our analysis it suffices to assume that the bulk spherical model is critical, so that $\xi_L\to\infty$ when $L\to\infty$. We will consider all the three critical regimes: (*a*) $L/\xi_L\to 0$; (*b*) $L/\xi_L=O(1)$, and (*c*) $L/\xi_L\to\infty$. At the final stage, see section 5, we discuss the sets of boundary conditions and dimensionalities which lead to one or the other alternative types of behaviour of $L/\xi_L$, as well as the geometrical origin of the logarithmic in $L$ corrections.

Obviously, terms proportional to $\ln L$ in the free energy can arise directly from powers of $L$ in the product $\tilde{Q}_{d,d'}(\xi)$, see equations (2.9) and (2.12), or from powers of $\xi$ when $\xi_L=O(L)$ as $L\to\infty$. A less obvious source of $\ln L$ terms are the factors asymptotically proportional to powers of $\exp(L/\xi)$ with $L/\xi\propto\ln L$ as $L\to\infty$. Indeed, by using standard finite-size analysis it can be shown [10] that at $d=3$ the presence of free surfaces in the spherical model leads to a positive logarithmic shift in the critical coupling

$$K_{c,L}=K_c+a\ln L/L \tag{4.1}$$

where $a>0$ is a geometry-dependent amplitude. As a result, in an infinitesimal neighbourhood of the shifted critical coupling one has

$$\xi_L(K)=O(L) \qquad (K-K_{c,L})L^{1/\nu}=O(1) \tag{4.2}$$

but in an infinitesimal neighbourhood of the bulk critical coupling ($d=3$)

$$\xi_L(K)=L/b\ln L \qquad (K-K_c)L^{1/\nu}=O(1) \tag{4.3}$$

where $b>0$ is a geometry-dependent amplitude. Hence, in the regime (4.3) $L/\xi_L=b\ln L$.

Now we pass to a finite-size analysis of the right-hand side of equation (3.9) with the aim to select all the terms asymptotically proportional to $\ln L$, $\ln\xi$ and $L/\xi$.

First of all, by using the explicit expressions (3.3) and (3.4) we note that if $a\to 0$ as $L\to\infty$, then

$$\ln g_L^{(0)}(a)=\begin{cases} 2\ln(aL)+O(a^2) & aL\to 0 \qquad (4.4a)\\ 2\ln[2\sinh(aL/2)]+O(a^2) & aL=O(1) \qquad (4.4b)\\ aL[1+O(a^2)]+O(\mathrm{e}^{-aL}) & aL\to\infty. \qquad (4.4c)\end{cases}$$

$$\ln g_L^{(1)}(a)=\begin{cases} \ln(aL)+O(a^2) & aL\to 0 \qquad (4.5a)\\ \ln[\sinh(aL)]+O(a^2) & aL=O(1) \qquad (4.5b)\\ aL[1+O(a^2)]+O(\mathrm{e}^{-2aL}) & aL\to\infty \qquad (4.5c)\end{cases}$$

Hence, the terms $\ln Q_{1,0}(\xi)$ (see (3.6)) and $\ln Q_{1,1}(\xi)$ (see (3.5)) yield

$$\ln Q_{1,0}(\xi)\cong\begin{cases} \ln L & L/\xi\to 0 \qquad (4.6a)\\ \ln L & L/\xi=O(1) \qquad (4.6b)\\ \ln\xi+L/\xi & L/\xi\to\infty \qquad (4.6c)\end{cases}$$

$$\ln Q_{1,1}(\xi)\cong\begin{cases} 2\ln L & L/\xi\to 0 \qquad (4.7a)\\ 2\ln L & L/\xi=O(1) \qquad (4.7b)\\ 2\ln\xi+L/\xi & L/\xi\to\infty. \qquad (4.7c)\end{cases}$$

Next we consider the contributions from $\ln R_{n,m}^{(\tau)}(\xi)$, $\tau=0, 1$ (see (3.8))

$$\ln R_{n,m}^{(\tau)}(\xi)=\sum_{\nu\in s}\sum_{k_\nu=1}^{L-1}\ln g_L^{(\tau)}([\xi^{-2}+\lambda_{n,m}(k)]^{1/2}). \tag{4.8}$$

Since for the arguments $a=[\xi^{-2}+\lambda_{n,m}(k)]^{1/2}$ of the function $\ln g_L^{(\tau)}(a)$ in the right-hand side of (4.8) we have the uniform in $k_\nu=1,\ldots,L-1$, $\nu=1,\ldots,n$, estimates ($n\geqslant 1$):

$$aL=L[\xi^{-2}+\lambda_{n,m}(k)]^{1/2}\geqslant\left[L^2/\xi^2+2L^2\left(1-\cos\frac{\pi}{L}\right)\right]^{1/2}\approx[L^2/\xi^2+\pi^2]^{1/2} \tag{4.9a}$$

and

$$aL\leqslant L[\xi^{-2}+4n]^{1/2} \tag{4.9b}$$

from (4.4) and (4.5), it follows that if $L/\xi\to 0$, or $L/\xi=O(1)$

$$\ln R_{n,m}^{(\tau)}(\xi)=O(L^{n+1})\times(\text{analytical in } \xi^{-2} \text{ function})$$
$$+O(L^n)\times(\text{analytical in } L^2/\xi^2 \text{ function}). \tag{4.10}$$

Finally, to study the regime $L/\xi\to\infty$, when $\xi\to\infty$ and $L\to\infty$, we introduce the notation

$$A=S\cap D' \qquad B=S\cap(D\backslash D') \qquad |S|=n \qquad |A|=m \tag{4.11}$$

and rewrite (4.8) in the form

$$\ln R_{n,m}^{(\tau)}(\xi)=S_{n,m}^{(\tau)}(\xi;\varepsilon,L)+\sum_{\mu\in A}\sum_{k_\mu\geqslant\varepsilon L}^{(1-\varepsilon)L}\sum_{\nu\in B}\sum_{k_\nu\geqslant\varepsilon L}^{L-1}\ln g_L^{(\tau)}([\xi^{-2}+\lambda_{n,m}(k)]^{1/2}) \tag{4.12}$$

with some small but fixed $\varepsilon>0$. For the arguments $a=[\xi^{-2}+\lambda_{n,m}(k)]^{1/2}$ of the function $\ln g_L^{(\tau)}(a)$ in the second term in the right-hand side of (4.12) we have the estimate

$$aL\geqslant[L^2/\xi^2+2L^2(1-\cos\pi\varepsilon)]]^{1/2}\cong L[\xi^{-2}+\pi^2\varepsilon^2]^{1/2} \tag{4.13}$$

which proves that the considered term is analytical in $\xi^{-2}$ if $\xi\to\infty$ as $L\to\infty$. Therefore, contributions proportional to $L/\xi$ as $L\to\infty$ may appear from

$$S_{n,m}^{(\tau)}(\xi;\varepsilon,L)=2^m\sum_{\nu\in s}\sum_{k_\nu=1}^{\varepsilon L}\ln g_L^{(\tau)}([\xi^{-2}+\lambda_{n,m}(k)]^{1/2}). \tag{4.14}$$

Since the arguments of the function $\ln g_L^{(\tau)}(a)$ in this case do not exceed a value of order $O(\varepsilon)$, we can use the expansion (4.4$c$) and (4.5$c$) to obtain the leading-order asymptotic form

$$S_{n,m}^{(\tau)}(\xi;\varepsilon,L)\cong 2^m L\sum_{\nu\in S}\sum_{k_\nu=1}^{\varepsilon L}\left[\xi^{-2}+\sum_{\mu\in A}(2\pi k_\mu/L)^2+\sum_{\nu\in B}(\pi k_\mu/L)^2\right]^{1/2}. \tag{4.15}$$

Consider first the case $|A|=0$ and $|B|=1$. Then, by adding and subtracting the $k=0$ term and using the Poisson summation formula [11], we obtain

$$S^{(\tau)}_{1,0}(\xi;\varepsilon,L)\cong L\sum_{k=1}^{\varepsilon L}[\xi^{-2}+(\pi k/L)^2]^{1/2}$$

$$=L\sum_{q=-\infty}^{\infty}\int_0^{\varepsilon L}\mathrm{d}p\,\mathrm{e}^{\mathrm{i}2\pi pq}[\xi^{-2}+(\pi p/L)^2]^{1/2}-\tfrac{1}{2}L/\xi$$

$$=L^2I(\xi;\varepsilon,0)+2L^2\sum_{q=1}^{\infty}I(\xi;\varepsilon,Lq)-\frac{1}{2}L/\xi \tag{4.16}$$

where

$$I(\xi;\varepsilon,Lq)=\frac{1}{\pi}\int_0^{\varepsilon}\mathrm{d}x\cos(2Lqx)\,(\xi^{-2}+x^2)^{1/2} \tag{4.17}$$

A direct evaluation of the integral (4.17) with $q=0$ gives

$$I(\xi;\varepsilon,0)=\frac{\varepsilon}{2}(\varepsilon^2\pi^2+\xi^{-2})^{1/2}+\frac{1}{2\pi}\xi^{-2}\ln[\varepsilon\pi+(\varepsilon^2\pi^2+\xi^{-2})^{1/2}]+\frac{1}{2\pi}\xi^{-2}\ln\xi. \tag{4.18}$$

For $I(\xi;\varepsilon,Lq)$ with $q\neq 0$ integration by parts under the assumption that $\varepsilon L$ is an integer and $L/\xi\to\infty$ gives

$$I(\xi;\varepsilon,Lq)=\pi^{-1}(2Lq)^{-2}[\varepsilon\pi(\varepsilon^2\pi^2+\xi^{-2})^{-1/2}+\text{terms}\propto\exp(-2Lq/\xi)]. \tag{4.19}$$

Therefore

$$S^{(\tau)}_{1,0}(\xi;\varepsilon,L)\cong-\tfrac{1}{2}L/\varepsilon+\ldots \tag{4.20}$$

where the dots stand for contributions which are not proportional to $\ln\xi$ or $L/\xi$.

Consider next the case $|A|=1$ and $|B|=0$. Then

$$S^{(\tau)}_{1,1}(\xi;\varepsilon,L)\cong 2L\sum_{k=1}^{\varepsilon L}[\xi^{-2}+(2\pi k/L)^2]^{1/2}\cong 4S^{(\tau)}_{1,0}(2\xi;\varepsilon,L)\simeq -L/\xi+\ldots. \tag{4.21}$$

Further we note that if $|A|=0$ and $n=|B|>1$, one can write

$$S(\tau)_{n,0}(\xi,\varepsilon,L)\cong L\sum_{k_1=1}^{\varepsilon L}\cdots\sum_{k_n=1}^{\varepsilon L}\left[\xi^{-2}+\sum_{\nu=1}^{n}(\pi k_\nu/L)^2\right]^{1/2}=-\tfrac{1}{2}S^{(\tau)}_{n-1,0}(\xi;\varepsilon,L)$$

$$+L\sum_{k_2=1}^{\varepsilon L}\cdots\sum_{k_n=1}^{\varepsilon L}\sum_{q=-\infty}^{\infty}\int_0^{\varepsilon L}\mathrm{d}p\,\mathrm{e}^{\mathrm{i}2\pi pq}\left[\xi^{-2}+\sum_{\nu=2}^{n}(\pi k_\nu/L)^2+(\pi p/L)^2\right]^{1/2}. \tag{4.22}$$

The analysis of the integral in the right-hand side of (4.22) repeats the one already carried out for $n=1$, the only difference being that in equations (4.16)–(4.19) $\xi^{-2}$ should be replaced by $\xi^{-2}+\Sigma(\pi k_\nu/L)^2$. Thus we conclude that

$$S^{(\tau)}_{n,0}(\xi;\varepsilon,L)\cong-\tfrac{1}{2}S^{(\tau)}_{n-1,0}(\xi;\varepsilon,L)+\cdots=(-\tfrac{1}{2})^n L/\xi+\ldots \tag{4.23}$$

hence

$$\ln R^{(\tau)}_{n,0}(\xi)\cong(-\tfrac{1}{2})^n L/\xi+\ldots\ (L/\xi\to\infty). \tag{4.24}$$

Similarly, in the general case we find

$$S^{(\tau)}_{n,m}(\xi;\varepsilon,L)\cong(-1)^m S^{(\tau)}_{n-m,0}(\xi;\varepsilon,L)+\cdots=(-1)^m(-\tfrac{1}{2})^{n-m}L/\xi+\ldots \tag{4.25}$$

hence

$$\ln R^{(\tau)}_{n,m}(\xi)\cong(-1)^m(-\tfrac{1}{2})^{n-m}L/\xi+\ldots(L/\xi\to\infty) \tag{4.26}$$

which completes the first stage of our analysis.

Let us specify now the boundary conditions and consider the contributions $\Delta F_{d,d'}$ in the free energy (2.12) from the above considered factors.

(1) In the case of fully free boundary conditions, by combining equations (2.12), (3.9) at $d'=0$, (4.6) and: (i) (4.10) if $L/\xi_L\to 0$ or $L/\xi_L=O(1)$; (ii) (4.24) if $L/\xi_L\to\infty$, we obtain

$$\Delta F_{d,0}=\begin{cases}-2^{-d}\ln L+\ln(L/\xi_L) & L/\xi_L\to 0\\ -2^{-d}\ln L & L/\xi_L=O(1)\\ -2^{-d}\ln\xi_L+d2^{-d}L/\xi_L & L/\xi_L\to\infty.\end{cases} \tag{4.27}$$

Therefore, a logarithmic contribution appears in all cases when $\xi_L\to\infty$ as $L\to\infty$. Standard finite-size analysis of the mean spherical constraint [10] shows that the case $\xi_L\propto L$ takes place in the neighbourhood of the shifted critical temperature, and the case $\xi_L\propto L/\ln L$ as $L\to\infty$ is realized at the bulk critical point.

(2) In the case of fully periodic boundary conditions, by combining equations (2.12), (3.9) at $d'=d$, (4.7) and: (i) (4.10) if $L/\xi_L\to 0$ or $L/\xi_L=O(1)$; (ii) (4.26) if $L/\xi_L\to\infty$, we obtain

$$\Delta F_{d,d}=\begin{cases}\ln(L/\xi_L) & L/\xi_L\to 0\\ 0 & L/\xi_L=O(1)\\ 0 & L/\xi_L\to\infty.\end{cases} \tag{4.28}$$

Therefore, a logarithmic contribution appears only when $L/\xi_L\to 0$ as $L\to\infty$. Standard finite-size analysis of the mean spherical constraint shows that the case $L/\xi_L\to 0$ as $L\to\infty$ is realized at dimensionalities $d$ equal to, or higher than the upper critical one $d_u=4$. Then [12, 13]

$$\xi_L/L\propto\begin{cases}(\ln L)^{1/4} & d=4\\ L^{(d-4)/4} & d>4\end{cases} \tag{4.29}$$

which implies

$$\Delta F_{d,d}=\begin{cases}-\dfrac{1}{4}\ln\ln L & d=4\\ -\dfrac{d-4}{4}\ln L & d>4.\end{cases} \tag{4.30}$$

In the scaling regime, $2<d<4$, one has $L/\xi_L=O(1)$ and, therefore, no terms proportional to $\ln L$ appear in the free energy.

(3) In the case of free boundaries in $d-d'\geqslant 1$ dimensions and periodic boundaries in $d'\geqslant 1$ dimensions, by combining equations (2.12), (3.9), (4.6), (4.7) and: (i) (4.10) if $L/\xi_L\to 0$ or $L/\xi_L=O(1)$; (ii) (4.26) if $L/\xi_L\to\infty$, we obtain

$$\Delta F_{d,d'}=\begin{cases}\ln(L/\xi_L) & L/\xi_L\to 0\\ 0 & L/\xi_L=O(1)\end{cases} \tag{4.31}$$

and

$$\Delta F_{d,d'}=\begin{cases}2^{-d}L/\xi_L & L/\xi_L\to\infty \quad d'=1\\ 0 & L/\xi_L\to\infty \quad d'\geqslant 2.\end{cases} \tag{4.32}$$

Therefore, a logarithmic contribution may appear when $L/\xi_L\to 0$ as $L\to\infty$, or in the special case when $d'=1$ and $\xi_L\propto L/\ln L$ as $L\to\infty$. As already mentioned, standard finite-size analysis of the mean spherical constraint [10] shows that the case $\xi_L\propto L$ takes place in the neighbourhood of the shifted critical temperature, and the case $\xi_L\propto L/\ln L$ as $L\to\infty$ is realized at the bulk critical point.

## 5. Discussion

The mean spherical model is particularly interesting for the theory of phase transitions, since it exhibits a non-classical phase transition and permits the derivation of exact results for both the thermodynamic and the finite-size critical properties.

As mentioned in the introduction, the other known models for which analogous results are available are the Gaussian model in the critical regime $L/\xi\to 0$ [3, 4] and the constrained monomer-dimer (CMD) model [7]. However, the free energy of the Gaussian model diverges at criticality, $\xi\to\infty$, unless the zero-eigenvalue mode is removed by hand. In the CMD model this mode is absent, but logarithmic corrections still appear under periodic boundary conditions in $d'\geqslant 1$ dimensions.

We have shown, that unlike the above-mentioned models, the mean spherical model in the finite-size scaling regime $(K-K_{c,L})L^{1/\nu}=O(1)$, when $L/\xi_L=O(1)$, completely obeys the theoretical predictions: logarithmic in $L$ corrections stem only from the corners, see equations (4.27), (4.28) and (4.31). Since a $d$-dimensional system with fully free boundary conditions has $2^d$ corners, the contribution per corner is

$$\Delta F_d^{(\mathrm{corner})}(K\cong K_{c,L})=-2^{-2d}\ln L. \tag{5.1}$$

The free energy of the critical Gaussian model with the $k=0$ mode removed exhibits an unpredicted, geometry-independent $\ln L$ term under $d'\geqslant 1$ periodic boundaries. Only if the free energy difference $F^G_{d,0}-F^G_{d,d'}$ for some $d'\geqslant 1$ is considered, the contribution per corner (5.1) would follow. The same holds, up to the coefficient $-2$, for the CMD model.

An interesting result is the appearance of a $\ln\ln L$ term at the upper critical dimensionality, equation (4.30).

The $\ln L$ term persisting under fully periodic boundary conditions for all $d>d_u$ (4.30) can be cast in the form

$$\Delta F_{d,d}=-\frac{d-4}{4d}\ln N \qquad d>4. \tag{5.2}$$

Hence, by taking the limit $d\to\infty$ we obtain $\Delta F_{\infty,\infty}=-\frac{1}{4}\ln N$, which is exactly the logarithmic correction found in [14] for the free energy of the infinitely coordinated mean spherical model containing $N$ particles.

Some unexpected results are found for the mean spherical model with free boundaries at the bulk critical point $K=K_c$, when $L/\xi_L\to\infty$. By comparing equations (4.27) and (4.32) we may conclude that in the geometry with fully free boundaries the contribution per corner is, compare with (5.1),

$$\Delta F_d^{(\mathrm{corner})}(K_c)=-2^{-2d}\ln\xi_L(K_c) \tag{5.3}$$

and the contribution from each of the $d2^{d-1}$ edges is

$$\Delta F_d^{(\mathrm{edge})}(K_c)=2^{-2d+1}L/\xi_L(K_c). \tag{5.4}$$

Note that a $d$-dimensional system with one periodic boundary has $2^{d-1}$ edges, so that equation (4.32) gives the same form (5.4) for the contribution per edge.

Our anlaysis of the three-dimensional case [10] has shown that in the presence of free surfaces the correlation length $\xi_L$ at the bulk critical coupling $K=K_c$ gains a logarithmic factor ($d=3$):

$$L/\xi_L(K_c)=\begin{cases}3\ln L & d'=0\\ 2\ln L & d'=1\\ \ln L & d'=2.\end{cases} \tag{5.5}$$

By inserting (5.5) at $d'=0$ in (5.3) at $d=3$ we obtain the explicit corner contribution at the bulk critical point,

$$\Delta F_d^{(\mathrm{corner})}(K_c)=-\frac{1}{64}\ln L+\frac{1}{64}\ln\ln L \tag{5.6}$$

and similarly from (5.4) we find the corresponding edge contribution ($d'=0$)

$$\Delta F_{d,0}^{(\mathrm{edge})}(K_c)=\frac{3}{32}\ln L. \tag{5.7}$$

Note that the logarithmic edge contribution depends on the boundary conditions ($d'=0$ or 1) through the correlation length (5.5). Thus from (5.5) at $d'=1$ and (5.4) at $d=3$ we obtain

$$\Delta F_{d,1}^{(\mathrm{edge})}(K_c)=\frac{1}{16}\ln L. \tag{5.8}$$

To the best of our knowledge, double-logarithmic corrections from corners and logarithmic corrections from one-dimensional edges have not been predicted by finite-size scaling theory (see, for example, the recent paper [15]). Their appearance in the mean spherical model with free surfaces is due to the fact that the finite-size shift in the critical temperature modifies the asymptotic behaviour of the correlation length from the usual type, $\xi_L(K_{c,L})\propto L$, to the anomalous one, $\xi_L(K_c)\propto L/\ln L$. Thus, the edge contributions proportional to $L/\xi_L$ (equation (5.4)), become logarithmic in $L$ at the bulk critical point, and the corner contributions proportional to $\ln\xi_L$ (equation (5.3)), acquire additional double-logarithmic in $L$ corrections.

The above results may have general implications for the finite-size scaling theory, since they demonstrate the importance of the way in which criticality is approached.

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